

# MEASUREMENT OF VERY SMALL PHASE FLUCTUATIONS BY MEANS OF THE OPERATIONAL APPROACH

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Recently Noh, Fougères and Mandel (NFM) [1] have improved the operational approach to the quantum phase problem substantially and measured the phase dispersion of coherent light down to very small mean photon numbers of the order of  $10^{-2}$ . This has prompted many other investigations and clarified some important questions in relation to what is actually measured. Although their treatment is rather general, we confine ourselves here to the case of a strong local oscillator (LO) and reproduce their measurement scheme in Fig. 1. Surprisingly enough, this simultaneous measurement of the sine and the cosine of the phase difference is completely equivalent to an old proposal to measure the phase after strong linear amplification [2] realized experimentally by the Welling group [3]. The reason for this rests on the fact, that in both cases the results are determined by the  $Q$  function of the signal. This was shown for amplification in [4] and for the measurement after beam splitting by Lai and Haus [5] and also in [6, 7, 8]. The measured phase dispersion is given by

$$\begin{aligned} (\delta\varphi)^2 &= 1 - \left| \ll e^{i\varphi} \gg \right|^2 . \\ &= 1 - \left| \sum_{n=0}^{\infty} c_n c_{n+1}^* b_n^1 \right|^2 , \end{aligned} \quad (1)$$

where the double brackets mean a classical average over the radius integrated  $Q$  function of the signal and the  $b_n^1$  are defined by

$$b_n^1 = \frac{\Gamma(n + 1/2 + 1)}{\sqrt{n!(n+1)!}} . \quad (2)$$

These coefficients are all smaller than one and broaden therefore the phase distribution of the pure state  $|\psi\rangle$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle . \quad (3)$$

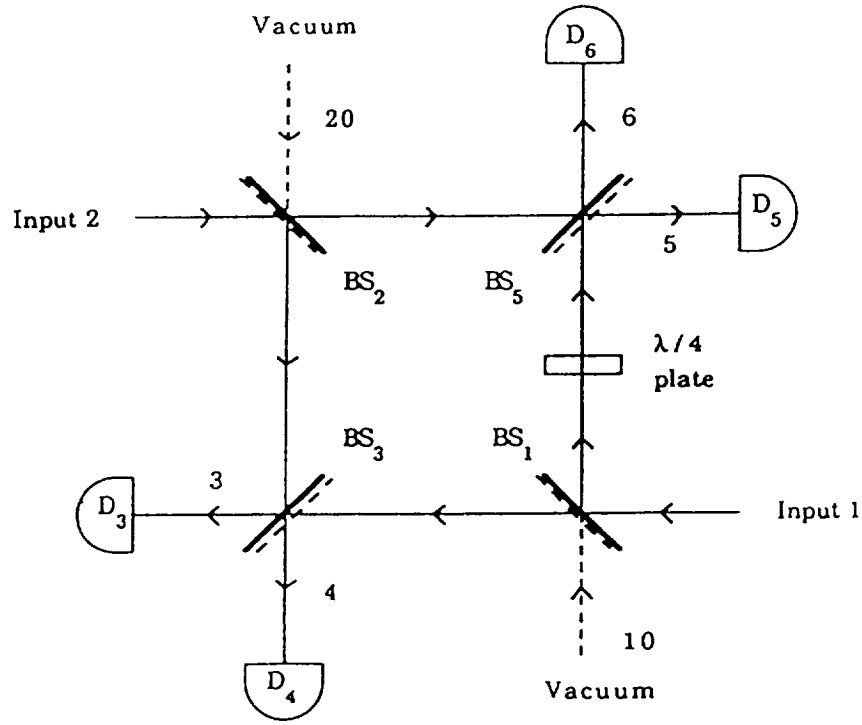
In showing that the  $b_n^1$  result from the calculation of the dispersion with the help of the NFM operators we found the expansion [8]

$$b_n^1 = 1 - \frac{1}{8(n+1)} + \frac{1}{128(n+1)^2} + \frac{5}{8 \cdot 128(n+1)^3} - + \dots , \quad (4)$$

that evidently proves the above mentioned property  $b_n^1 < 1$  and  $\lim_{n \rightarrow \infty} b_n^1 = 1$ . This expansion converges excellently and is very useful because eq. (1) reduces to the Pegg-Barnett (PB) dispersion [9] by putting all  $b_n^1$  equal to one, the zeroth approximation of eq. (4).

Now, very small phase fluctuations suppose great photon number fluctuations and the last can change the interference signal. Of such kind is the situation for states near to the so-called phase optimized states (POS) [10,11] which are characterized by the relation [11]

$$(\delta\varphi)_{PB}^2 = \frac{1 \cdot 89}{(N+1)^2}, \quad N \gg 1, \quad (5)$$



**Fig. 1** Outline of the experimental scheme used by Noh, Fougères and Mandel where the sine and the cosine of the phase difference are measured simultaneously.  $BS_i$  are identical 50/50 beam splitters and  $D_j$  are photodetectors. Input 2 is the local oscillator.

where  $N$  is the mean photon number. Note that coherent states lead to

$$(\delta\varphi)_{PB}^2 = \frac{1}{4N}, \quad N \gg 1. \quad (6)$$

For clarity the subscripts PB indicate that eqs. (5) and (6) are Pegg-Barnett dispersions i. e., they are calculated by replacing the classical average in eq. (1) by the quantum ensemble average  $\langle e^{i\hat{\varphi}} \rangle$  with the Hermitian phase operator  $\hat{\varphi}$  [9].

The question is now how to determine such small dispersions from measurements in the operational approach, i. e., by measuring via the radius integrated  $Q$  function (eqs. (1), (2) and (4)). The answer is that with the help of the expansion (4), some limitations and an additional measurement of the photon distribution in the scheme of Fig. 1 it is possible to infer the PB dispersion of states with a  $(\delta\varphi)^2$  comparable to the value of eq. (5). Note that the measurement of eq. (1) alone cannot give adequate information about phase dispersions near to POS. This is illustrated in Fig. 2 for two-photon coherent states (TCS) that can be optimized to come close to POS for certain degrees of squeezing  $s$  at a fixed mean photon number  $N$  [12].

The following investigations are rather analogous to calculations made by Ritze [13] in his different proposal to measure extremely small phase fluctuations. First, one has to find a suitable reference phase in order to make the  $c_n$  in eq. (1) real. This corresponds to  $\langle \sin \hat{\varphi} \rangle = 0$ , where we suppress the phase of the LO. Second, only such input fields can be admitted that allow a truncation of the expansion (4):

$$\ll \cos \varphi \gg_Q = \sum_{n=0}^{\infty} c_n c_{n+1} - \frac{1}{8} \sum_{n=0}^{\infty} \frac{c_n c_{n+1}}{n+1} + \frac{1}{128} \sum_{n=0}^{\infty} \frac{c_n c_{n+1}}{(n+1)^2}. \quad (7)$$

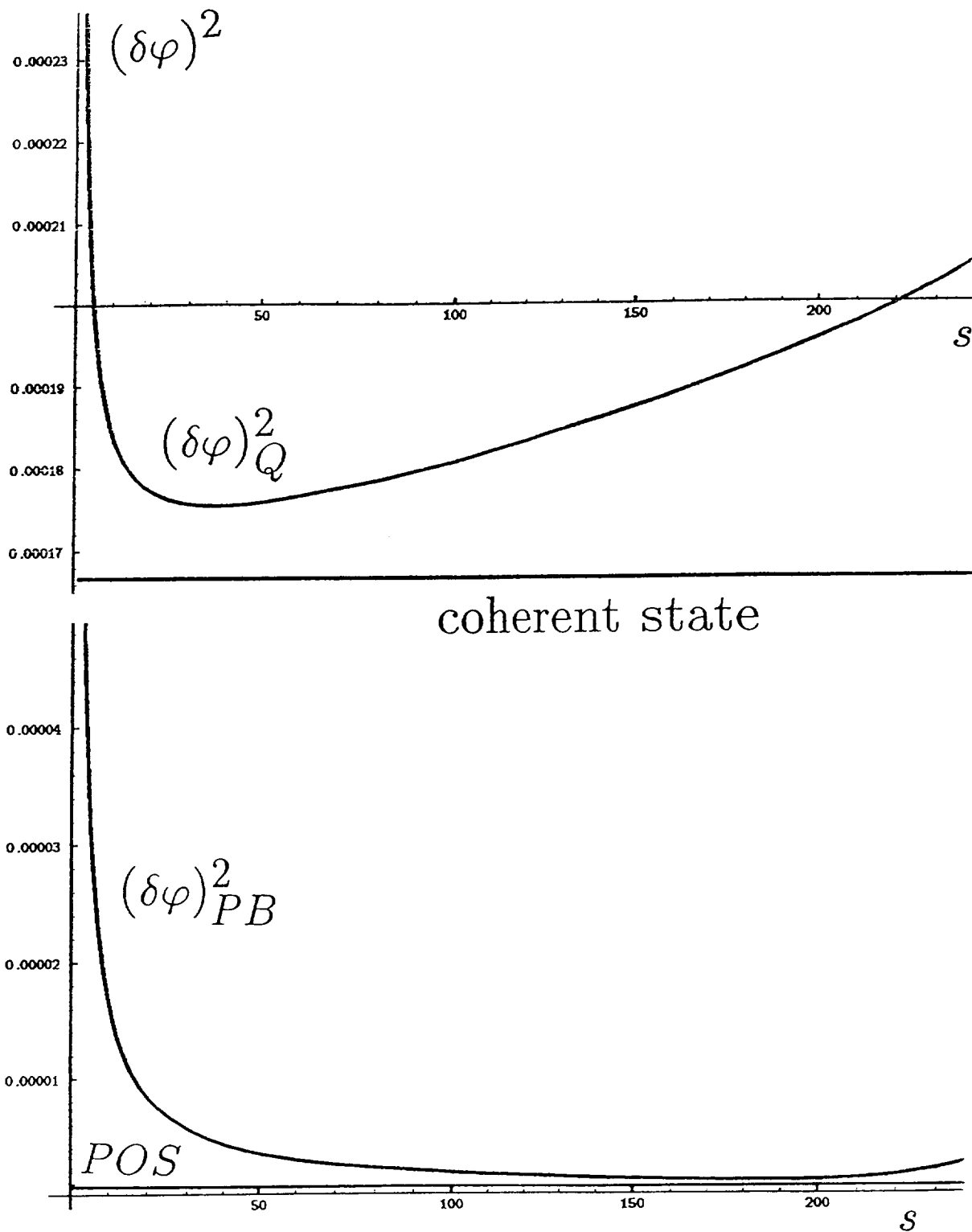


Fig. 2 Phase dispersions for TCS at a mean photon number of  $N = 1500$  and increasing squeezing parameter  $s$ . Note that  $s = 1$  describes coherent light. The  $Q$  function based dispersion  $(\delta\varphi)^2_Q$  starts at  $1/2N$ , comes close to the coherent state value  $1/4N$  and increases again. The Pegg-Barnett dispersion  $(\delta\varphi)^2_{PB}$  begins at  $1/4N$  for  $s = 1$ , decreases sharply and reaches its minimum near to the POS level at strongly different  $s$  values than  $(\delta\varphi)^2_Q$ .

Third, we assume a smooth  $c_n$  distribution and approximate the  $c_{n+1}$  by

$$c_{n+1} = c_n + c'_n, \quad (8)$$

where  $c'_n$  denotes the derivative of  $c_n$  with respect to  $n$ . With eq. (8) we can determine the second sum in eq. (7):

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{c_n c_{n+1}}{n+1} &\cong \sum_{n=0}^{\infty} \frac{c_n^2}{n+1} + \frac{1}{2} \int_0^{\infty} \frac{d(c_n^2)}{n+1} \\ &= \left\langle \frac{1}{\hat{n}+1} \right\rangle - \frac{1}{2} c_0^2 + \frac{1}{2} \int_0^{\infty} \frac{c_n^2}{(n+1)^2} dn, \end{aligned} \quad (9)$$

where  $\hat{n}$  is the photon number operator and  $\langle \dots \rangle$  the normal quantum average. Due to the Schwarz inequality we find in addition [14,2]

$$\left( \sum_{n=0}^{\infty} c_n c_{n+1} \right)^2 \leq \sum_{n=0}^{\infty} c_n^2 \sum_{n=0}^{\infty} c_{n+1}^2 \quad (10)$$

and therefore  $(\delta\varphi)_{PB}^2 \geq c_0^2$ . It turns out that POS fulfil  $(\delta\varphi)_{PB}^2 \gg c_0^2$  and our truncation assumption requires the same. Therefore  $c_0^2$  can be neglected in eq. (9) and we obtain eventually

$$\ll \cos \varphi \gg_Q = \ll \cos \varphi \gg_{PB} - \frac{1}{8} \left\langle \frac{1}{\hat{n}+1} \right\rangle - \frac{7}{128} \left\langle \frac{1}{(\hat{n}+1)^2} \right\rangle. \quad (11)$$

It is evident that for the determination of the averages over the number operator expressions on the right-hand side of eq. (11) we need the knowledge of the photon number distribution. This is not surprising because very small phase fluctuations require enhanced photon number fluctuations that affect the interference signal. During the measurement of such phase dispersions we must consequently also monitor the photon number fluctuations.

The situation changes remarkably, if we omit the first beam splitter in Fig. 1 and put the signal into each channel as illustrated in Fig. 3. This makes sense if we confine ourselves to two-photon coherent states (TCS) because then the radius integrated Wigner function is measured as was shown for coherent states by Freyberger and Schleich [6] and generalized to TCS by Leonhardt and Paul [15]. The measured dispersion is now

$$(\delta\varphi)_W^2 = 1 - \left| \sum_{n=0}^{\infty} c_n c_{n+1}^* A_n^1 \right|^2, \quad (12)$$

where the subscript  $W$  points to the Wigner function and the  $A_n^1$  can be expanded into the series

$$A_n^1 = 1 + \frac{(-1)^n}{4(n+1)} + \frac{1}{32(n+1)^2} - \frac{(-1)^n \cdot 5}{128(n+1)^3} + \dots \quad (13)$$

Eq. (13) shows clearly the oscillations about one what amounts to the fact that eqs. (12) and (6) give exactly corresponding results in this order ( $1/N$ ). However, for TCS's near to POS (eq. (5)), as introduced in [12], the next order,  $1/N^2$ , is dominating. Here, the term  $1/32(n+1)^2$  of the  $A_n^1$  plays an important role. The result is that the measured dispersion  $(\delta\varphi)_W^2$  can be smaller than the corresponding PB result. Thus the measurement following the scheme of Fig. 3 yields for coherent states with  $N \gg 1$  the Pegg-Barnett result while second-order effects can change the measured dispersion for optimized TCS drastically for moderate  $N$  ( $\cong 50$ ). For very large  $N$   $(\delta\varphi)_{PB}^2$  and  $(\delta\varphi)_W^2$  coincide for TCS [16].

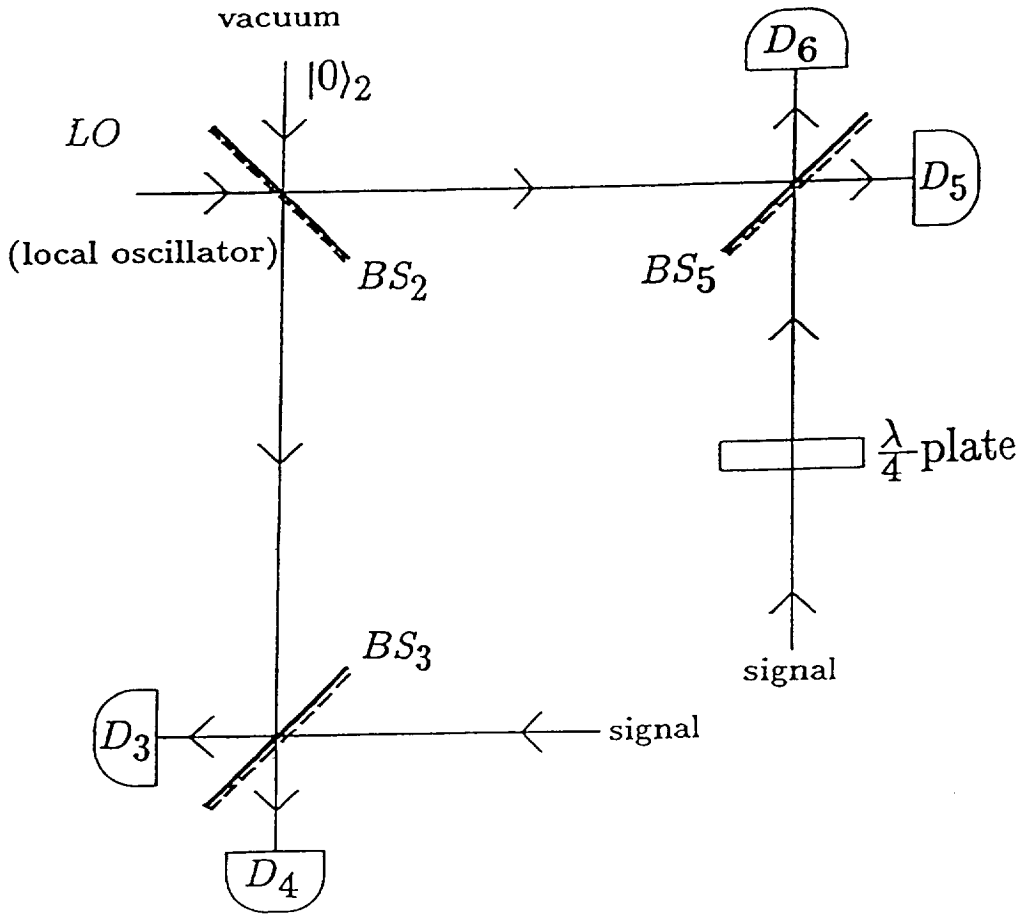


Fig. 3 Modified homodyne detection scheme with two input ports where two identically prepared signals are incident. The  $\lambda/4$ -plate makes, as in Fig. 1, a  $\pi/2$  phase shift in order to measure simultaneously the sine and the cosine of the phase difference. The signal is here not contaminated by the vacuum from the unused port. Thus, there is no physical reason for any broadening as in Fig. 1.

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